ON COMMON FIXED POINTS OF COMMUTING CONTINUOUS FUNCTIONS ON AN INTERVAL

BY
JOHN PHILIP HUNEKE

This paper offers two methods of constructing commuting pairs of continuous functions (i.e. f, g such that $f(g(x))=g(f(x))$) which map $[0, 1]$ to itself without common fixed points. Any such pair will be called "a solution to the commuting function problem".

PART I

LEMMA 1. *Let* $(f_n \mid n \in N)$, $(g_n \mid n \in N)$ *be two uniformly convergent sequences of continuous functions from* $[0, 1]$ *to itself with limits* f, g *respectively. If* $f_n g_{n+1} = g_n f_{n+1}$ *for each* $n \in N$, *then* f, g *commute.*

LEMMA 2. *Let* h *be a piecewise linear nowhere constant function* (*i.e. the derivative of* h, *denoted* Dh, *is nowhere zero*) *defined on* I; *and let* $A \subset h(I)$ *be a finite set which contains the image under* h *of all points in the interior of* I *at which* Dh *does not exist. If* r, s *are consecutive* (*with respect to the natural order on* R) *in* $h^{-1}(A)$, *then* $h(r), h(s)$ *are consecutive in* A.

PROPOSITION 1. *Let* $(f_n \mid n \in N)$, $(g_n \mid n \in N)$ *be sequences of piecewise linear functions from* $[0, 1]$ *to itself such that either* $|x - f_2(x)| > 1/6$ *or* $|x - g_2(x)| > 1/6$ *for each* $x \in [0, 1]$. *Then the limits of the sequences* $(f_n \mid n \in N)$, $(g_n \mid n \in N)$ *form a solution to the commuting function problem provided that there exists a sequence* $(A_n \mid n \in N)$ *of finite subsets of* $[0, 1]$, *and for each* $n \in N$, *the following properties are valid*:

$P_1(n)$: $f_n g_{n+1} = g_n f_{n+1}$;
$P_2(n)$: $|Df_n(x)| \geqq 3$ *and* $|Dg_n(y)| \geqq 3$ *wherever the derivatives exist*;
$P_3(n)$: *if either* $Df_n(x)$ *or* $Dg_n(x)$ *does not exist, then* $x \in A_n$;
$P_4(n)$: *for each pair of consecutive points* r, s *of* A_{n+1} *with* $r < s$,

$$f_n([r, s]) = f_{n+1}([r, s]) \quad \text{and} \quad g_n([r, s]) = g_{n+1}([r, s]);$$

$P_5(n)$: $f_n^{-1}(A_n) = A_{n+1} = g_n^{-1}(A_n)$; *and*
$P_6(n)$: $0, 1 \in A_n$.

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Proof. For any $m \in N^+$ and for any pair of consecutive points r, s of A_{m+1}, there exists by $P_5(n)$ and by Lemma 2 a pair of consecutive points r', s' of A_m (equal to $g_m(r), g_m(s)$ respectively) such that $\frac{1}{3}|f_{m-1}(r')-f_{m-1}(s')| \geqq |f_m(r)-f_m(s)|$ since:

$$\begin{aligned}
\tfrac{1}{3}\cdot|f_{m-1}(r')-f_{m-1}(s')| &= \tfrac{1}{3}\cdot|f_{m-1}(g_m(r))-f_{m-1}(g_m(s))| \\
&= \tfrac{1}{3}\cdot|g_{m-1}(f_m(r))-g_{m-1}(f_m(s))| \qquad [\text{by } P_1(m-1)] \\
&= \tfrac{1}{3}\cdot|f_m(r)-f_m(s)|\cdot\frac{|g_{m-1}(f_m(r))-g_{m-1}(f_m(s))|}{|f_m(r)-f_m(s)|} \\
&\geqq \tfrac{1}{3}|f_m(r)-f_m(s)|\cdot 3 \qquad [\text{by } P_5(m), \text{ Lemma 2, } P_3(m-1), \text{ and } P_2(m-1)] \\
&= |f_m(r)-f_m(s)|.
\end{aligned}$$

For every $n \in N$ and for every $x \in [0, 1]$, there exist two consecutive points r, s of A_{n+1} such that $r \leqq x \leqq s$ (since A_{n+1} is finite and contains 0, 1 by $P_6(n+1)$), and so it follows that $|f_n(x)-f_{n+1}(x)| \leqq |f_n(r)-f_n(s)| \leqq (1/3)^n$ by $P_4(n)$, $P_3(n)$, and by iterated use of the result established in the preceding sentence. Therefore $\|f_n-f_{n+1}\| \leqq 3^{-n}$. Since $(f_n \mid n \in N)$, $(g_n \mid n \in N)$ have symmetric roles, we also have for each $n \in N$ $\|g_n-g_{n+1}\| \leqq 3^{-n}$.

Now, $(f_n \mid n \in N)$, $(g_n \mid n \in N)$ are uniformly convergent sequences, since for every pair $n, m \in N$ with $n \leqq m$, we have:

$$\|f_n-f_m\| \leqq \sum_{i=n}^{+\infty} \|f_i-f_{i+1}\| \leqq \sum_{i=n}^{+\infty} 3^{-i} = (2\cdot 3^{n-1})^{-1};$$

and similarly $\|g_n-g_m\| \leqq (2\cdot 3^{n-1})^{-1}$. From these inequalities, the limits of the two sequences of functions have no common fixed point since $\|f_2-\lim_{n\to\infty} f_n\| \leqq 1/6$ and $\|g_2-\lim_{n\to\infty} g_n\| \leqq 1/6$. Also, the limits commute by Lemma 1.

NOTATION. If f is a piecewise linear function defined on I, then let $B(f)$ denote the set of all points in the interior of I at which f has no derivative (i.e. at which Df does not exist).

LEMMA 3. *Let f be a piecewise linear function defined on I; let g be a function defined on I; let h be a linear function with range I; and let 1 be a nonconstant linear function defined on $f(I)$. Then $gh(B(fh))=g(B(f))$, $B(1f)=B(f)$, and $h^{-1}(B(f))=B(f(h))$.*

DEFINITION. For any piecewise linear function f, if there exists an $s \in R^+$ such that $|Df(x)|=s$ for each x for which $Df(x)$ exists, then f has a *derivative of constant absolute value* which we will denote by f has DCAV; also, we will denote s by *slope f*.

NOTATION. If I is an interval with endpoints r, s, then let $T_{rs}: I \to I$ denote the linear function defined for each $x \in I$ by $T_{rs}(x)=r+s-x$.

PROPOSITION 2. *Let f: $[a, a'] \to [b, b']$ and g: $[c, c'] \to [b, b']$ be piecewise linear functions. If $\{f(a), f(a')\}=\{b, b'\}=\{g(c), g(c')\}$ and if f, g have DCAV, then there* *exist piecewise linear functions* $j\colon [d, d'] \to [a, a']$ *and* $k\colon [d, d'] \to [c, c']$ *satsifying the properties*: $j(d)=a$, $j(d')=a'$; $\{k(d), k(d')\}=\{c, c'\}$; j, k *have DCAV*; $j(B(k)) \subset B(f)$, $k(B(j)) \subset B(g)$; *and* $fj=gk$. *The following diagram commutes*:

$$\begin{tikzcd} & {[a, a']} \arrow[dr, "f"] & \\ {[d, d']} \arrow[ur, "j"] \arrow[dr, "k"'] & & {[b, b']} \\ & {[c, c']} \arrow[ur, "g"'] & \end{tikzcd}$$

Proof. First Proposition 2 will be proved under the additional assumption that $f(a)=b=g(c)$ (and hence: $f(a')=b'=g(c')$). We proceed by induction on the number of distinct points in $B(f) \cup B(g)$. Pick an interval $[d, d']$.

If $B(f) \cup B(g)$ is empty, then f and g are linear, so define j, k to be the unique linear maps on $[d, d']$ such that $j(d)=a$, $j(d')=a'$, $k(d)=c$, and $k(d')=c'$. Observe that fj and gk are linear functions on $[d, d']$ such that $fj(d)=b=gk(d)$ and $fj(d')=gk(d')$. Therefore $fj=gk$; also $B(j)$ and $B(k)$ are empty. So j, k satisfy Proposition 2.

Now let $n \in N$ and assume the induction hypothesis: Proposition 2 is valid whenever $f(a)=b=g(c)$ and $B(f) \cup B(g)$ has no more than n elements. Suppose we are given f, g satisfying the hypothesis of Proposition 2 such that $f(a)=b=g(c)$ and $B(f) \cup B(g)$ has $n+1$ elements. Since $B(f) \cup B(g)$ is finite and nonempty, let b'' denote the smallest element of $f(B(f)) \cup g(B(g))$. Since f and g play symmetric roles, assume without loss of generality that $b'' \in f(B(f))$. Let a'' denote any point in $B(f)$ such that $f(a'')=b''$. $B(f)$ equals the set of points which are local minimum or local maximum points for f except for a, a', because f has DVAC; also, a'' is a local minimum for f since f has a local maximum at the endpoint a' of $[a, a']$. Now f has local minima at a and a'', so f must have a local maximum between a and a'', i.e. $B(f|_{[a,a'']})$ is nonempty. Let b''' denote the maximum element of $f(B(f|_{[a,a'']}))$, and let a''' be any element of $B(f|_{[a,a'']})$ such that $f(a''')=b'''$. Let c''' be the minimum element of $[c, c']$ such that $g(c''')=b'''$, and let c'' be the maximum element of $[c, c''']$ such that $g(c'')=b''$. Let $d''=(2d+d')/3$ and let $d'''=(d+2d')/3$. Observe that the induction hypothesis is applicable to each of the three pairs of functions: (i) $f|_{[a,a''']}$, $g|_{[c,c''']}$; (ii) $fT_{a''a'''}$, $g|_{[c'',c''']}$; and (iii) $f|_{[a'',a']}$, $g|_{[c'',c']}$. Therefore, for (i) there exists j', k' defined on $[d, d'']$, for (ii) there exists j'', k'' defined on $[d'', d''']$, and for (iii) there exists j''', k''' defined on $[d''', d']$ satisfying Proposition 2 (as stated for f, g, j, k respectively). See Figure 1.

Now define the two points d^*, d^{**}:

$$d^* = d+(d'-d)\cdot\frac{(\operatorname{slope} j')}{(\operatorname{slope} j' + \operatorname{slope} j'' + \operatorname{slope} j''')};$$

$$d^{**} = d+(d'-d)\cdot\frac{(\operatorname{slope} j' + \operatorname{slope} j'')}{(\operatorname{slope} j' + \operatorname{slope} j'' + \operatorname{slope} j''')}.$$

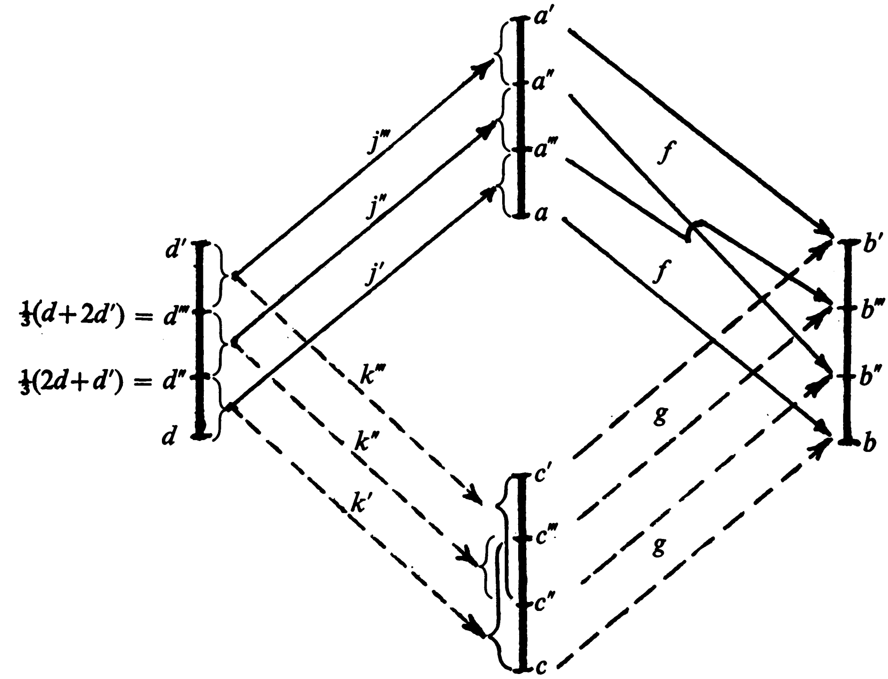

FIGURE 1. Induction step in the proof of Proposition 2.

Let S' be the linear function defined from $[d, d^*]$ onto $[d, d'']$ such that $S'(d)=d$, let S'' be the linear function defined from $[d^*, d^{**}]$ onto $[d'', d''']$ such that $S''(d^*) = d'''$, and let S''' be the linear function defined from $[d^{**}, d']$ onto $[d''', d']$ such that $S'''(d')=d'$. Let $j=j'S' \cup T_{a''a'''}j''S'' \cup j'''S'''$ (considering a function to be a set of ordered pairs), and let $k=k'S' \cup k''S'' \cup k'''S'''$. By checking the points d^*, d^{**}, it can be seen that j, k are well defined functions (i.e. $j(d^*)=a'''$, $j(d^{**})=a''$, $k(d^*)=c'''$, and $k(d^{**})=c''$). It is clear that j, k are piecewise linear functions. Also j, k have DCAV with

$$\text{slope}\, j=\tfrac{1}{3}(\text{slope}\, j' + \text{slope}\, j'' + \text{slope}\, j''') \text{ and slope}\, k=\tfrac{1}{3}(\text{slope}\, k' + \text{slope}\, k'' + \text{slope}\, k''').$$

Observe that $j(d)=a$, $j(d')=a'$, $k(d)=c$, and $k(d')=c'$. We also have:

$$\begin{aligned} j(B(k)) &\subset \{j(d^*), j(d^{**})\} \cup j(B(k'S')) \cup j(B(k''S'')) \cup j(B(k'''S''')) \\ &= \{a'', a'''\} \cup j'S'(B(k'S')) \cup T_{a''a'''}j''S''(B(k''S'')) \cup j'''S'''(B(k'''S''')) \\ &= \{a'', a'''\} \cup j'(B(k')) \cup T_{a''a'''}j''(B(k'')) \cup j'''(B(k''')) \quad \text{[by Lemma 3]} \\ &\subset \{a'', a'''\} \cup B(f|_{[a,a''']}) \cup B(fT_{a''a'''}T_{a''a'''}^{-1}) \cup B(f|_{[a'',a']}) \\ &= B(f). \end{aligned}$$

Therefore, $j(B(k)) \subset B(f)$. Now, since j has DCAV, $x \in B(j)$ only if x is either a local maximum or a local minimum of j. Neither d^* nor d^{**} is a local maximum or a local minimum of j, so neither is in $B(j)$. So:

$$\begin{aligned} k(B(j)) &= k(B(j'S')) \cup k(B(T_{a''a'''}j''S'')) \cup k(B(j'''S''')) \\ &= k'S'(B(j'S')) \cup k''S''(B(T_{a''a'''}j''S'')) \cup k'''S'''(B(j'''S''')) \\ &= k'(B(j')) \cup k''(B(j'')) \cup k'''(B(j''')) \qquad \text{[by Lemma 3]} \\ &\subset B(g|_{[c,c''']}) \cup B(g|_{[c'',c''']}) \cup B(g|_{[c'',c']}) \\ &\subset B(g). \end{aligned}$$

Therefore, $k(B(j)) \subset B(g)$. It is clear that $fj = gk$. Hence, j, k satisfy Proposition 2. By induction, Proposition 2 has been proved provided $f(a) = b = g(c)$.

Now assume that f, g satisfy the hypothesis of Proposition 2.

CASE 1. If $f(a) = b = g(c)$, then j, k exist satisfying Proposition 2 as defined above.

CASE 2. If $f(a) = b = g(c')$, then apply Case 1 to f, $gT_{cc'}$ to get two functions j, k; now j, $T_{cc'}k$ satisfy Proposition 2.

CASE 3. If $f(a) = b' = g(c')$, then apply Case 1 to $fT_{aa'}$, g to get two functions j, k; now $T_{aa'}jT_{dd'}$, $kT_{dd'}$ satisfy Proposition 2.

CASE 4. If $f(a) = b' = g(c)$, then apply Case 1 to $fT_{aa'}$, $gT_{cc'}$ to get two functions j, k; now $T_{aa'}jT_{dd'}$, $T_{cc'}kT_{dd'}$ satisfy Proposition 2.

This completes the proof of Proposition 2.

LEMMA 4. *Let the two functions h, k map $[a, a']$ onto $[b, b']$ and let h be linear and k be piecewise linear. If k has DCAV, then* slope $h \leqq$ slope k.

THE CONSTRUCTION. We are now ready to construct (using Proposition 2) sequences $(f_n \mid n \in N)$, $(g_n \mid n \in N)$, $(A_n \mid n \in N)$ which satisfy the conditions of Proposition 1. Specifically we desire two sequences $(f_n \mid n \in N)$, $(g_n \mid n \in N)$ of piecewise linear functions mapping $[0, 1]$ into itself such that for every $x \in [0, 1]$ either $|x - f_2(x)| > \frac{1}{6}$ or $|x - g_2(x)| > \frac{1}{6}$, and a sequence $(A_n \mid n \in N)$ of finite subsets of $[0, 1]$ such that $0, 1 \in A_0$, and such that the following properties are satisfied for every $n \in N$:

$P_i(n)$: for $i = 1, 2, 3, 4, 5$ as in Proposition 1;

$P_7(n)$: $f_n|_{A_{n+1}} = f_{n+1}|_{A_{n+1}}$ and $g_n|_{A_{n+1}} = g_{n+1}|_{A_{n+1}}$;

$P_8(n)$: $f_n(B(g_n)) \cup g_n(B(f_n)) \subset A_n$;

$P_9(n)$: $A_n \subset A_{n+1}$; and,

$P_{10}(n)$: for all r', s' consecutive in A_n such that $r' < s'$, $f_n|_{[r',s']}$ and $g_n|_{[r',s']}$ each have DCAV.

Now define $A_0 = \{0, 1\}$, $A_1 = \{0, 1/3, 2/3, 1\}$, and

$$A_2 = \{0, 1/9, 2/9, 1/3, 6/15, 7/15, 8/15, 9/15, 2/3, 7/9, 8/9, 1\}.$$

Also, define f_0, g_0, f_1, g_1, f_2, and g_2 at each $x \in [0, 1]$ as follows:

if $0 \leqq x < 1/3$, let $f_0(x) = 3x$, and let $g_0(x) = 1 - 3x$;

if $1/3 \leqq x < 2/3$, let $f_0(x) = 2 - 3x$, and let $g_0(x) = 3x - 1$;

if $2/3 \leqq x \leqq 1$, let $f_0(x)=3x-2$, and let $g_0(x)=3-3x$;
if $0 \leqq x < 1/3$ or $2/3 \leqq x \leqq 1$, let $f_1(x)=f_0(x)$, and let $g_1(x)=g_0(x)$;
if $1/3 \leqq x < 6/15$, let $f_1(x)=(8/3)-5x$;
if $6/15 \leqq x < 7/15$, let $f_1(x)=5x-(4/3)$;
if $7/15 \leqq x < 2/3$, let $f_1(x)=(10/3)-5x$;
if $1/3 \leqq x < 8/15$, let $g_1(x)=5x-(5/3)$;
if $8/15 \leqq x < 9/15$, let $g_1(x)=(11/3)-5x$;
if $9/15 \leqq x < 2/3$, let $g_1(x)=5x-(7/3)$;
if $0 \leqq x < 6/15$ or $7/15 \leqq x \leqq 1$, let $f_2(x)=f_1(x)$;
if $6/15 \leqq x < 31/75$, let $f_2(x)=(25/3)x-(8/3)$;
if $31/75 \leqq x < 32/75$, let $f_2(x)=(38/9)-(25/3)x$;
if $32/75 \leqq x < 7/15$, let $f_2(x)=(25/3)x-(26/9)$;
if $0 \leqq x < 8/15$ or $9/15 \leqq x \leqq 1$, let $g_2(x)=g_1(x)$;
if $8/15 \leqq x < 41/75$, let $g_2(x)=(49/9)-(25/3)x$;
if $41/75 \leqq x < 42/75$, let $g_2(x)=(25/3)x-(11/3)$; and
if $42/75 \leqq x < 9/15$, let $g_2(x)=(17/3)-(25/3)x$.

Observe that f_i, g_i, A_i for $i=0, 1, 2$ have been defined satisfying the desired properties. By Figure 2 for each $x \in [0, 1]$, $|x-f_2(x)| > 1/6$ or $|x-g_2(x)| > 1/6$. Also, the properties $P_1(i-1)$, $P_2(i)$, $P_3(i-1)$, $P_4(i-1)$, $P_5(i-1)$, $P_7(i-1)$, $P_8(i)$, $P_9(i-1)$, and $P_{10}(i)$ are satisfied for $i=1, 2$.

Proceed to define $(f_n \mid n \in N)$, $(g_n \mid n \in N)$, and $(A_n \mid n \in N)$ by induction; let $n \in N$, $n \geqq 2$, and assume that f_i, g_i, and A_i (for $i=0, 1, 2, \ldots, n$) have been defined satisfying the desired properties, especially: $P_1(i-1)$, $P_2(i)$, $P_3(i-1)$, $P_4(i-1)$, $P_5(i-1)$, $P_7(i-1)$, $P_8(i)$, $P_9(i-1)$, and $P_{10}(i)$ for $i=1, 2, \ldots, n$. Define $A_{n+1} = f_n^{-1}(A_n)$. Notice that

$$A_{n+1} = f_n^{-1}(A_n) = f_n^{-1}g_{n-1}^{-1}(A_{n-1}) = g_n^{-1}f_{n-1}^{-1}(A_{n-1}) = g_n^{-1}(A_n),$$

so $P_5(n)$ is satisfied. For each $x \in [0, 1]$ define $f_{n+1}(x)$ and $g_{n+1}(x)$ in the following manner. There exist two consecutive points d, d' of A_{n+1} such that $d \leqq x \leqq d'$ (since A_{n+1} is finite and contains 0, 1). We can set $\{a, a'\}=\{g_n(d), g_n(d')\}$, $\{b, b'\} =\{f_{n-1}g_n(d), f_{n-1}g_n(d')\}$, and $\{c, c'\}=\{f_n(d), f_n(d')\}$ such that $a<a'$, $b<b'$, and $c<c'$ by Lemma 2. Now observe that $f_n|_{[a,a']}$, $g_n|_{[c,c']}$ satisfy the hypothesis of Proposition 2 for f, g respectively; so let j, k be defined by Proposition 2, j mapping $[d, d']$ onto $[a, a']$ and k mapping $[d, d']$ onto $[c, c']$. If $g_n(d)=a$, then define $f_{n+1}(x)=k(x)$, and $g_{n+1}(x)=j(x)$; if $g_n(d)=a'$, then define $f_{n+1}(x)=kT_{dd'}(x)$, and $g_{n+1}(x)=jT_{dd'}(x)$. In the above procedure, for a fixed $x \in [0, 1]$, the points d, d' (defined to be consecutive points in A_{n+1} such that $d \leqq x \leqq d'$) are not necessarily unique. However, f_{n+1} and g_{n+1} are well defined functions since $g_{n+1}|_{A_{n+1}}=g_n|_{A_{n+1}}$ by definition, and hence $f_{n+1}|_{A_{n+1}}=f_n|_{A_{n+1}}$. Therefore $P_7(n)$ is satisfied. $P_1(n)$ is satisfied since for each x in $[0, 1]$ there exist d, d' consecutive in A_{n+1} such that $d \leqq x \leqq d'$ and either $f_ng_{n+1}(x)=f_njT_{dd'}(x)=g_nkT_{dd'}(x)=g_nf_{n+1}(x)$, or $f_ng_{n+1}(x)=f_nj(x)= g_nk(x)=g_nf_{n+1}(x)$ where j, k are defined by Proposition 2 as above; hence f_ng_{n+1}

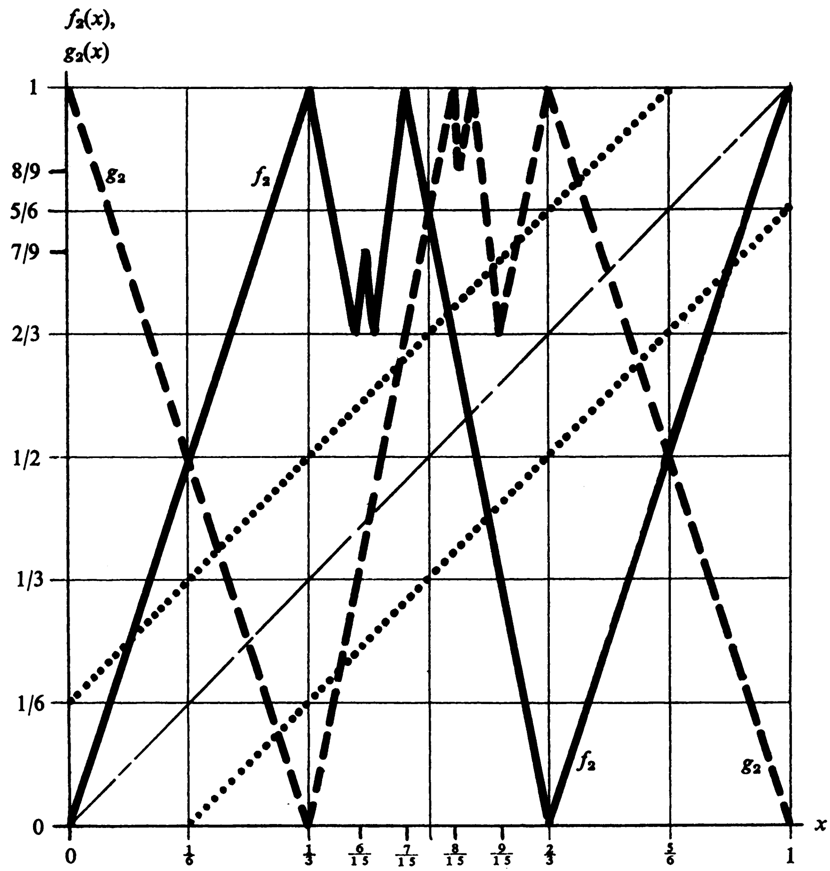

FIGURE 2. The graphs of f_2, g_2 and the diagonal; f_2 or g_2 lies between the dotted lines wherever $|x-f_2(x)|\leqq\frac{1}{6}$ or $|x-g_2(x)|\leqq\frac{1}{6}$ respectively.

$=g_nf_{n+1}$. $P_3(n)$ is satisfied by $P_5(n)$ and $P_8(n)$. $P_4(n)$ is satisfied by $P_3(n)$ and Proposition 2. $P_{10}(n+1)$ is satisfied by Proposition 2. $P_2(n+1)$ is satisfied by $P_2(n)$, $P_3(n)$, $P_4(n)$, $P_{10}(n+1)$, and Lemma 4. $P_9(n)$ is satisfied by $P_5(n-1)$, $P_7(n-1)$, $P_9(n-1)$, and $P_5(n)$. $P_8(n+1)$ is satisfied by Proposition 2, Lemma 3, $P_5(n)$, $P_9(n)$, and $P_3(n)$. Therefore f_i, g_i, and A_i (for $i=0, 1, 2, \ldots, n, n+1$) have been defined satisfying the desired properties, especially: $P_1(i-1)$, $P_2(i)$, $P_3(i-1)$, $P_4(i-1)$, $P_5(i-1)$, $P_7(i-1)$, $P_8(i)$, $P_9(i-1)$, and $P_{10}(i)$ for $i=1, 2, \ldots, n, n+1$.

Therefore, the sequences $(f_n \mid n\in N)$, $(g_n \mid n\in N)$, and $(A_n \mid n\in N)$ have been defined by induction satisfying Proposition 1. So $\lim_{n\to\infty} f_n$ and $\lim_{n\to\infty} g_n$ form a solution to the commuting function problem.

REMARK. Simultaneous to and independent of the author's preceding work(1), W. M. Boyce [1], [2] constructed essentially the same solution to the commuting

(1) Compare [1] and [3].

function problem defined above. These functions are nowhere Lipschitzian; smoother solutions are described in Part II below.

Part II

Notation. For any real valued mapping h defined on a subset of the reals, let h^* denote the map: $h^*(x)=1-h(1-x)$ for each x for which $h(1-x)$ is defined; also, h will be called s-Lipschitzian provided s is a real number and for each x, y in the domain of h, $|h(x)-h(y)| \leqq s \cdot |x-y|$. Now pick any b in $[0, \frac{1}{2}]$, and define

$$s = \frac{3-2b+(6-4b)^{1/2}}{1-2b}.$$

Define the three linear functions:

h_1: $[b, (1-b+sb)/s] \to [b, 1]$ by $h_1(x)=sx-sb+b$;

h_2: $[(1-b+sb)/s, (2-b+sb)/s] \to [0, 1]$ by $h_2(x)=2-sx+sb-b$;

h_3: $[(2-b+sb)/s, (3-2b+sb)/s] \to [0, 1-b]$ by $h_3(x)=-2+sx-sb+b$.

And define the piecewise linear function h: $[b, (3-2b+sb)/s] \to [0, 1]$ by $h=h_1 \cup h_2 \cup h_3$. Let C_b denote all continuous functions from $[0, b]$ to $[0, b]$ which have b as a fixed point.

Definition. For each g in C_b let $\bar{g}$ denote the unique extension of g defined by:

1. $\bar{g}(x)=g(x)$ whenever $0 \leqq x \leqq b$;
2. $\bar{g}(x)=h(x)$ whenever $b \leqq x \leqq h_3^{-1}(1-b)$;
3. $\bar{g}(x)=h_1^{*-1}\bar{g}(h^*(x))$ whenever $h_3^{-1}(1-b) \leqq x \leqq h_2^{*-1}(h_2^{-1}(0))$;
4. $\bar{g}(x)=h_2^{*-1}\bar{g}(h^*(x))$ whenever $h_2^{*-1}(h_2^{-1}(0)) \leqq x \leqq 1-b$; and
5. $\bar{g}(x)=$ the fixed point of h_2^* whenever $1-b \leqq x \leqq 1$.

See Figure 3.

Remark. That the preceding definition is consistent can be checked by direct mechanical methods, or (as suggested by Felix Albrecht) by a Zorn's Lemma argument. The following sketch of a proof that $\bar{g}$ is uniquely defined above for any s-Lipschitzian g in C_b was suggested by David Boyd.

Proof. Let g be an s-Lipschitzian function in C_b, and let L denote the set of s-Lipschitzian functions from $[0, 1]$ to itself which satisfy properties 1, 2, and 5 for $\bar{g}$ in the Definition. Now define the mapping T from L to L: for any f in L, let

$T(f)(x)=f(x)$ whenever $0 \leqq x \leqq h_3^{-1}(1-b)$ or whenever $1-b \leqq x \leqq 1$;

$T(f)(x)=h_1^{*-1}f(h^*(x))$ whenever $h_3^{-1}(1-b) \leqq x \leqq h_2^{*-1}(h_2^{-1}(0))$;

$T(f)(x)=h_2^{*-1}f(h^*(x))$ whenever $h_2^{*-1}(h_2^{-1}(0)) \leqq x \leqq 1-b$.

To see that $T(f)$ is in L, observe that $T(f)$ is s-Lipschitzian by a system of inequalities using the facts that h_1^{*-1} and h_2^{*-1} are linear and $1/s$-Lipschitzian, $h_1^{*-1}(0) =h_2^{*-1}(0)$, h^* is s-Lipschitzian, and the fixed points of h^* are $h_3^{-1}(1-b)$, $(s-1)(1-b)/(s+1)$, and $1-b$.

L is clearly a complete metric space with respect to the supremum norm metric, and T is a contraction of this metric space with constant $1/s$ (since h_1^{*-1} and h_2^{*-1}

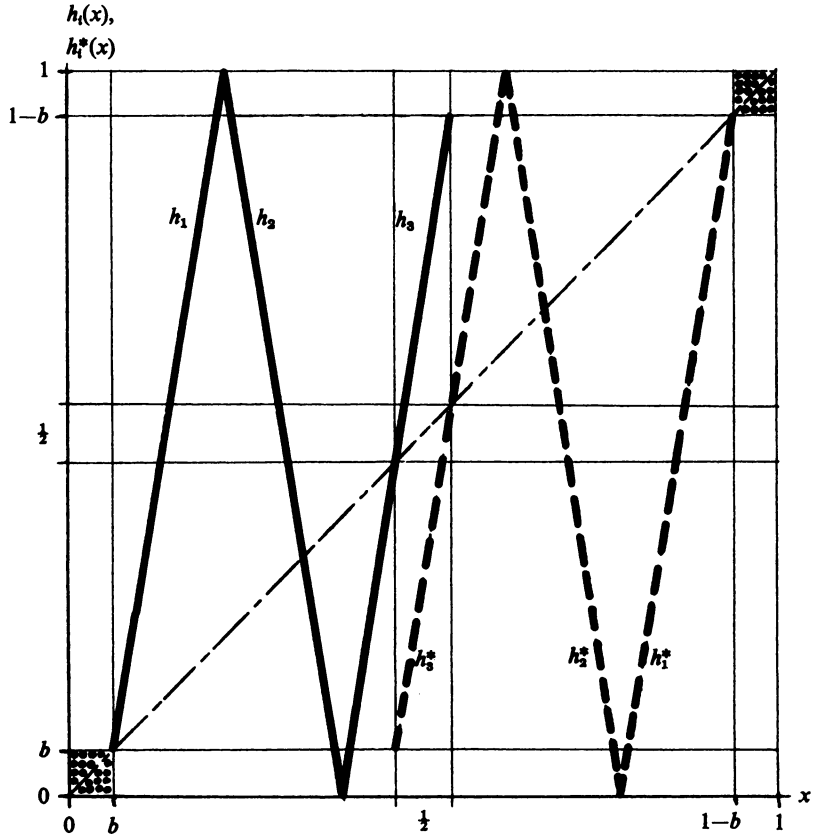

FIGURE 3. Graphs of h_i, h_i^* ($i=1, 2, 3$) with $b=3/50$ and $s=6$. The solid line is the graph of h; the dashed line is the graph of h^*. The dotted regions contain the graphs of g and g^* for any g in C_b.

are $1/s$-Lipschitzian). Hence there is a unique function $\bar{g}$ in L such that $T(\bar{g})=\bar{g}$, and this function satisfies the definition of $\bar{g}$.

LEMMA 5. *For any f, g in C_b and any x in* $[\frac{1}{2}, 1]$,
(1) $\bar{f}^*(x)=x$ *implies* $\bar{g}(x)\neq x$, *and*
(2) $\bar{f}^*(\bar{g}(x))=\bar{g}(\bar{f}^*(x))$.

Proof of (1). As a comment to clarify notation, $\bar{f}^*(x)=(\bar{f})^*(x)$. Observe that the domain of definition of h^* is $[h_3^{*-1}(b), 1-b]$, and

$$h_3^{*-1}(b) = \frac{-3+2b+s-sb}{s} = 1-b+\frac{(1-2b)(2b-3)}{3-2b+6-4b} < \frac{1}{2},$$

since $0\leq b<\frac{1}{2}$. So on $[\frac{1}{2}, 1]$, the fixed points of $\bar{f}^*$ are either in $[1-b, 1]$ or are the fixed points of h^*. By definition of $\bar{g}$ on $[1-b, 1]$, $\bar{g}(x)$ equals the fixed point of h_2^* which equals $(1-b)(s-1)/(1+s)<1-b\leq x$ for each $x\in[1-b, 1]$. Therefore $\bar{g}$

and $\bar{f}^*$ have no common fixed point in $[1-b, 1]$. The only fixed points of h^* are the fixed points of h_i^* for $i=1, 2, 3$. The fixed point of h_1^* is $1-b$, which (as has just been seen) is not a fixed point of $\bar{g}$. Denote the fixed point of h_2^* by x_2;

$$x_2 = \frac{(1-b)(s-1)}{s+1} < \frac{b-2+s+s^2-s^2b-2s}{s^2} = h_2^{*-1}(h_2^{-1}(0)),$$

and so

$$\bar{g}(x_2) \in h_1^{*-1}([0, 1]) = [(s-1+b-sb)/s, (s+b-sb)/s].$$

However,

$$x_2 = \frac{(1-b)(s-1)}{1+s} < \frac{s-1+b-sb}{s} \leqq \bar{g}(x_2),$$

so x_2, the fixed point of h_2^*, is not a fixed point for $\bar{g}$. Let x_3 denote the fixed point of h_3^*;

$$x_3 = \frac{3-s+sb-b}{1-s} = \frac{3-2b+sb}{s} = h_3^{-1}(1-b).$$

Therefore $\bar{g}(x_3)=g(h_3^{-1}(1-b))=1-b>x_3$. Therefore $\bar{f}^*$ and $\bar{g}$ have no common fixed point in $[\frac{1}{2}, 1]$.

Proof of (2). For each $x \in [1-b, 1]$, $\bar{f}^*(\bar{g}(x))=\bar{f}^*(x_2)=h_2^*(x_2)=x_2$, and $\bar{g}(\bar{f}^*(x)) =\bar{g}(1-\bar{f}(1-x))=\bar{g}(1-f(1-x))=x_2$ since $f(1-x) \in [0, b]$. Therefore $\bar{f}^*(\bar{g}(x)) =\bar{g}(\bar{f}^*(x))$ for each $x \in [1-b, 1]$. For each $x \in [h_2^{*-1}(h_2^{-1}(0)), 1-b]$,

$$\bar{f}^*(\bar{g}(x)) = \bar{f}^*(h_2^{*-1}\bar{g}(h^*(x))) = h_2^*h_2^{*-1}\bar{g}h^*(x) = \bar{g}\bar{f}^*(x);$$

hence, $\bar{f}^*(\bar{g}(x))=\bar{g}(\bar{f}^*(x))$. For each $x \in [h_3^{-1}(1-b), h_2^{*-1}(h_2^{-1}(0))]$,

$$\bar{f}^*(\bar{g}(x)) = \bar{f}^*(h_1^{*-1}\bar{g}(h^*(x))) = h_1^*h_1^{*-1}\bar{g}(h^*(x)) = \bar{g}(\bar{f}^*(x));$$

hence $\bar{f}^*(\bar{g}(x))=\bar{g}(\bar{f}^*(x))$. Now $h_3^{-1}(1-b)$ is the fixed point of h_3^* (as has been seen above), so

$$h_3^*([h_3^{*-1}(b), h_3^{-1}(1-b)]) = [b, h_3^{-1}(1-b)]$$

which equals the domain of definition of h; also $h_3^{*-1}(1-b)$ is the fixed point of h_3, so

$$h_3([h_3^{*-1}(b), h_3^{-1}(1-b)]) = [h_3^{*-1}(b), 1-b]$$

which equals the domain of definition of h^*. Observe that the two piecewise linear functions

$$h^*(h|_{[h_3^{*-1}(b), h_3^{-1}(1-b)]}), \qquad (h^*(h|_{[h_3^{*-1}(b), h_3^{-1}(1-b)]}))^*$$

are each the union of three linear functions and map the points $h_3^{*-1}(b)$, $h_3^{-1}(h^{*-1}(1))$, $h_3^{-1}(h^{*-1}(0))$, $h_3^{-1}(1-b)$ to b, 1, 0, $1-b$ respectively; hence the two functions coincide. Therefore for each $x \in [\frac{1}{2}, h_3^{-1}(1-b)]$, $x \in [h_3^{*-1}(b), h_3^{-1}(1-b)]$, so

$$\begin{aligned}\bar{f}^*(\bar{g}(x)) &= \bar{f}^*(h_3(x)) = h^*h(x) = (h^*h)^*(x) = 1-h^*(h(1-x)) \\ &= h(1-h(1-x)) = h(h^*(x)) = g(h_3^*(x)) = \bar{g}(\bar{f}^*(x)).\end{aligned}$$

Therefore $\bar{f}^*$ and $\bar{g}$ commute on $[\frac{1}{2}, 1]$ and have no common fixed point in $[\frac{1}{2}, 1]$.

PROPOSITION 3. *For any* f, g *in* C_b, $\bar{f}$ *and* $\bar{g}^*$ *form a solution to the commuting function problem.*

Proof. Let f, g be in C_b. Then by Lemma 5, $\bar{f}$ and $\bar{g}^*$ commute without common fixed point on $[\frac{1}{2}, 1]$; also $\bar{f}^*$ and $\bar{g}$ commute without common fixed point on $[\frac{1}{2}, 1]$. Therefore $\bar{f}^{**}$ and $\bar{g}^*$ commute without common fixed point on $[0, \frac{1}{2}]$. But $\bar{f}^{**} = \bar{f}$, so $\bar{f}$ and $\bar{g}^*$ form a solution to the commuting function problem.

COROLLARY. *If* f, g *are in* C_b, *then*:

(1) $\bar{f}$, $\bar{g}^*$ *form a solution to the commuting function problem*;

(2) $\bar{f}$, $\bar{g}^*$ *are s-Lipschitzian if and only if* f, g *are s-Lipschitzian*;

(3) $\bar{f}$, $\bar{g}^*$ *are linear on each component of a dense open subset of* $[0, 1]$ *if and only if* f, g *are linear on each component of a dense open subset of* $[0, b]$; *and*

(4) $\bar{f}$, $\bar{g}^*$ *are differentiable almost everywhere if and only if* f, g *are differentiable almost everywhere.*

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WESLEYAN UNIVERSITY,
MIDDLETÓWN, CONNECTICUT
UNIVERSITY OF MINNESOTA,
MINNEAPOLIS, MINNESOTA